Journal of Structural Geology, Vol. 9, No. 3, pp. 365 to 370, 1987
Printed in Great Britain

0191–8141/87 \$03.00 + 0.00

Practical section drawing through folded layers using sequentially rotated cubic interpolators

ANGUS M. MCCOSS

Department of Geology, The Queen's University, Belfast BT7 1NN, U.K.

(*Received 25 February 1986; accepted in revised form 20 September 1986*)

Abstract—A new, simple and practical cubic interpolation method, for use in microcomputer-aided section drawing, is presented. Apparent dips at pairs of consecutive points are used to determine the cubic interpolator between them. The process is repeated sequentially to construct the layer trace. Overturned layers are accommodated by rotating the reference frame of the interpolator. The extra degree of freedom introduced by rotating can be constrained using known axial traces, borehole data, or layer thicknesses. Unlike the Busk construction (arcs and tangents) the fold class demanded by the data can be conserved in the interpolation; that is, the technique is not restricted to parallel folds. In the absence of constraining data the reference frame can be rotated so that the ordinate parallels the bisector of consecutive dips, producing a conservative interpolation even where layers are inverted. Alternatively the rotated cubic with the minimum arc-length can be sought, providing an objective 'minimum' strain estimate in bed-length balancing.

INTRODUCTION

CROSS-SECTIONS are useful for analysing structure above and below the topographic surface. Recently, more and more techniques have used them as a starting point for further interpretation, for example in section balancing (see e.g. Dahlstrom 1969) and depth to decollement estimates. Apart from the Busk construction (Busk 1929), which is restricted to parallel folds, there are no repeatable and conservative methods in common usage for constructing cross-sections from dispersed data; indeed, most are drawn free-hand!

In the other sciences systematic interpolation is widely used to constrain interpretations. Amongst the methods commonly used is the technique of cubic spline interpolation, introduced by Schoenberg (1946). It has, however, been used sparingly by geologists and the mathematical complexity of the method requires use of a computer. A more important limitation to its use in section drawing is that the basic technique demands that values along the horizontal x-axis must continually increase, thus precluding interpolation of the closed and convolute curves common in geology. This problem can be overcome by using parametric cubic splines (Evans *et al.* 1985). However, this technique is only of use where the data comprise frequent digitised points along a fold profile, and not isolated apparent dips in the section plane.

An alternative method, which is more suited to section data and overcomes the problem of convolutions, simply involves sequentially rotating the reference frame prior to interpolation. This simplifies the mathematics greatly, replacing the large matrix operations of cubic spline interpolation (for discussion see Marchuk 1982 and Johnson & Riess 1982), by the use of planar rotations and subsequent substitution into a few relatively simple equations. The calculations can be rapidly executed using a microcomputer or even a programmable calculator. This paper attempts to present this new and simple approach to objective section drawing using sequentially rotated cubic interpolators in a clear algorithm which is tested using real and synthetic data.

BASIC CUBIC SPLINE INTERPOLATION

Cubic spline interpolation in its basic form permits a smooth curve to be drawn through a string of discrete data points lying in a plane. Each segment of the curve between adjacent points is part of a cubic, a third degree polynomial of the form

$$z = ax^3 + bx^2 + cx + d.$$

(Unless otherwise stated the z-axis is vertical and the x-axis is horizontal.) A cubic is generally selected in basic spline interpolation for three reasons. Firstly, it is the lowest order polynomial where adjacent curved segments merge with continuous first and second derivatives. Secondly, higher orders need more computing time and can have spurious 'extra' turning points between the data. Thirdly, a tentative physical reason for using cubics is that they approximate the form adopted for the minimization of strain energy in an elastic rod (or spline—hence the name).

An important limitation of the basic method is that successive points must have increasing values of x. This is perhaps one of the reasons why geologists have not fully exploited their usefulness, since, unmodified, the method cannot be used for complex cross-sections containing closed and convolute curves, with multiple values of z for a given value of x.

PARAMETRIC CUBIC SPLINE INTERPOLATION

Evans *et al.* (1985) advocate the use of parametric cubic splines to overcome the problem of multiple values of z in geological curves. Their technique involves using the continually increasing linear distance between successively digitised points on a known natural curve as a parameter against which the change in both x and z can be measured and interpolated separately. Although very useful for digitising layer traces from photographs or cut specimens, the technique cannot be used in unmodified form where the data comprise relatively isolated apparent dips along the topographic trace. Even when adapted to utilise slopes at isolated points (by rotating the interpolation ordinate so it is normal to the chord between consecutive points) the results generally lie well outside the range of subjectively acceptable profiles. This is largely due to the strong influence of the orientation of the chord between consecutive points, which in a geological situation is more likely to be a function of erosion than of folding. Fortunately this data format, using slopes at isolated points, can be used in other ways while permitting the mathematics to be greatly simplified.

ROTATED CUBIC INTERPOLATORS THROUGH KNOWN SLOPES

In a given reference frame there is a unique cubic segment which can be drawn through two tangent points, but the orientation of the reference frame controls the form of the interpolated curve. Figure 1 shows four such curves and the orientation of w, a vector parallel to the ordinate axes of the interpolation. The procedure for constructing the curves is outlined in a general algorithm for determining points on the interpolating rotated cubic arc (Appendix 1). From Fig. 1 it is clear that the chosen orientation of w is critical in determining the form and length of the arc. Special orientations with interesting properties are now considered and their limitations and merits discussed.

Vertical interpolation ordinates

In most scientific applications where slopes at points are known, polynomial interpolation would proceed using vertical ordinates. However, this does not permit the construction of curves with multiple z-values for a given value of x, as commonly occurs in folded layers. If one attempts this type of interpolation for apparent dip

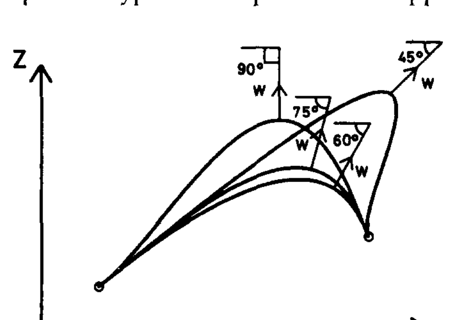

Fig. 1. Rotated cubic interpolators produced by rotation of w, a vector parallel to the interpolation ordinate, through 90, 75, 60 and 45°. Arc-length and morphology are strongly dependent on the orientation of w. Slopes are specified at the two tangent points (open circles).

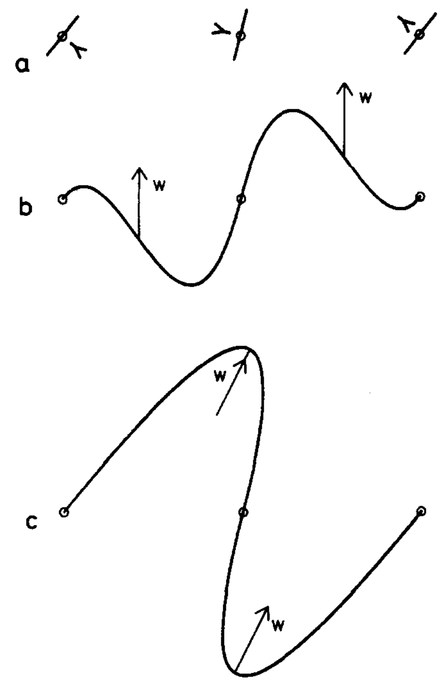

Fig. 2. (a) Slopes at three points (open circles) generate the curve in (b) when w is vertical and that in (c) when w parallels the acute bisector of consecutive slopes. The curve in (c) is more conservative in morphology than that in (b) which has an extra fold pair and a violation of younging direction at the central control point.

data from overturned layers, as shown in Fig. 2, an extra fold pair is generated and younging direction is not conserved along the length of the interpolated trace. This is obviously unacceptable.

Bisecting interpolation ordinates

By rotating w to parallel the bisector of consecutive apparent dips, a very useful and realistic interpolation is generated. Using the data from Fig. 2(a), the cubic interpolation shown in Fig. 2(c) results when w is chosen to parallel the acute bisector. The objective choice between acute or obtuse bisector is simply the one which conserves younging direction along the interpolation. This is readily achieved by using the bisector which parallels the sum of the two unit younging vectors, $(Y_1 + Y_2)$, (see Fig. 3). In the case depicted in Fig. 3(b),

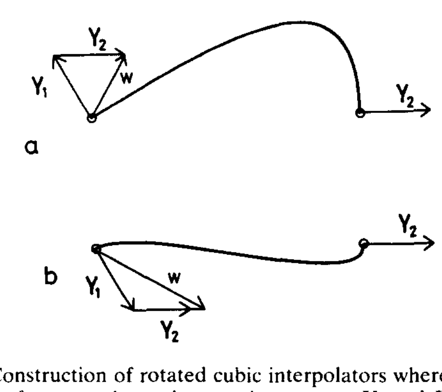

Fig. 3. Construction of rotated cubic interpolators where w parallels the sum of consecutive unit younging vectors Y_1 and Y_2. Facing is conserved along the length of the curves. In (a) the resulting orientation of w parallels the acute bisector of consecutive dips, whereas in (b) it parallels the obtuse bisector. The same pair of tangent points (open circles) is used in both (a) and (b).

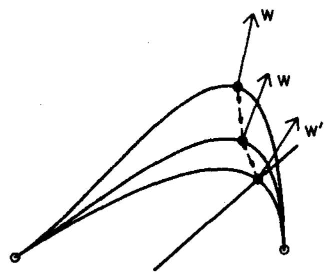

Fig. 4. Cubic interpolators produced by rotation of w. The point of maximum curvature (dot) migrates along the line of small arrows. The critical orientation of w (w') is selected when this point lies on the known axial trace (heavy line). Slopes are specified at two points (open circles).

if the two exposures are thought (from field evidence) to be linked continuously by a folded layer trace in the section, then data from the intermediate limb is obviously missing (due to poor exposure, for example).

Interpolation compatible with axial-trace data

If the position and orientation of the axial trace in the section are known, then the point of maximum curvature (c_{max}) of the interpolated curve can be migrated by rotating w until c_{max} lies on the axial trace, i.e. when $w = w'$ (Fig. 4). Although the axial trace is often subparallel to w' the two are not synonymous since one is a real line through hinges of folded layers whilst the other is an ordinate axis of an interpolating curve for a single layer trace. Adjacent layers need not share the same w', but will share the same axial trace. If only an orientation of the axial trace can be specified, however (e.g. from stereographic orientation analysis), then a good approximation of the layer trace can be constructed by rotating w to parallel it.

Interpolation compatible with borehole data

Interpolation can be further constrained using borehole or outcrop data from which the location of the layer is known, but no orientation is attainable. Where such data are available the curve should be migrated by rotating w until it passes through the point where the layer trace is known to occur. Figure 5 depicts the migration of the curve until it passes through a specific

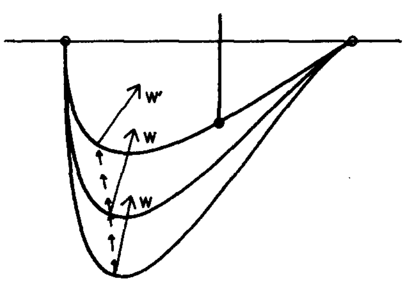

Fig. 5. Cubic interpolators produced by rotation of w. The critical orientation of w (w') is selected when a specific point (dot) down a borehole (heavy line) lies on the curve. Slopes are specified at two points (open circles).

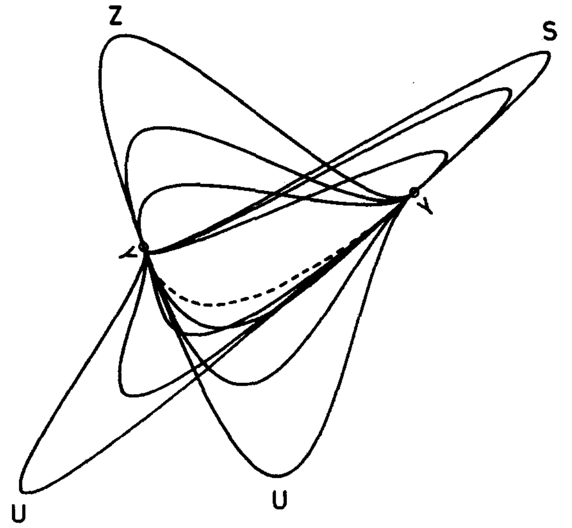

Fig. 6. Cubic interpolators produced by rotating w in steps through 180°. This produces a four-lobed pattern, two lobes of 'U-shaped' curves, one of 'S-shaped' curves and another of 'Z-shaped' curves. All satisfy the given slopes, but only the 'U-shaped' curves conserve the specified facing along their lengths. There is one critical 'U-shaped curve' which has minimum arc length (dashed). This critical minimum arc-length curve is in fact a rotated quadratic.

point down a borehole. If a few points are known to lie on the curve, but no information about their slopes is available, then the best curve could be selected by a least-squares approximation. If most of the data comprises such discrete points then the parametric method advocated by Evans *et al.* (1985) would be more appropriate.

Minimum arc-length interpolation

If w is rotated through the range θ = 0–180°, and the cubic arcs determined, then a pattern similar to that in Fig. 6 is produced. Only the 'U-shaped' arcs conserve the given younging directions along their lengths and of them there is one special one (dashed) which has the shortest arc-length. The critical w for a minimum arc-length (w_{min}) is not necessarily parallel to that which bisects the data, although is usually close to it. To determine the critical value of θ, it is necessary to sweep through values of θ near that of the bisector, computing to stage 5a of the algorithm (see Appendix 1), since the length of a 'U-shaped' arc tends to its minimum as the first coefficient, a, tends to zero. In fact the cubic is reduced to a rotated quadratic in this critical orientation. Once the critical value of θ has been determined, the rest of the algorithm is executed to construct the curve. The significance of this interpolation technique is that in the absence of extra information, a very conservative curve is constructed, which has a minimum cubic arc-length. Hence when combined with length-balancing a 'minimum shortening' estimate is produced.

Interpolation compatible with fold class data

If it is possible to determine the class of folds (e.g. using the t'_α and ϕ_α plots of Ramsay 1967 and Hudleston

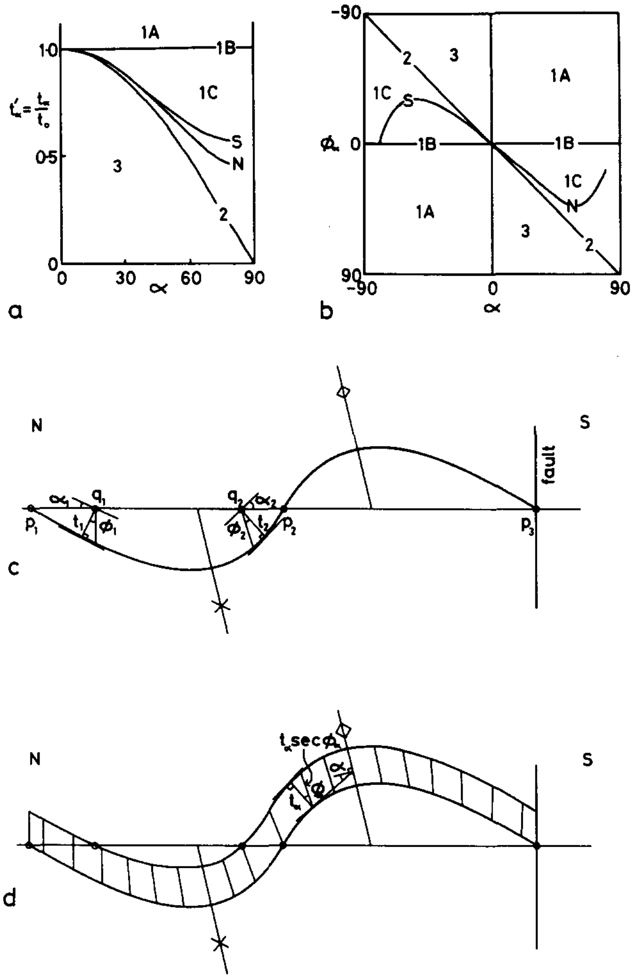

Fig. 7. Curves of (a) t'_α and (b) ϕ_α for a regionally typical pair of N–S-facing fold limbs (taken from Hudleston 1973). (c) Known slopes at p_1, p_2 and p_3 are used to construct sequentially rotated cubic interpolators which are compatible with the specified axial traces. This curve is chosen as a 'reference-curve' as it has the most control. A fault (heavy vertical line) prevents the slope of a curve passing through the points q_1 and q_2 from being determined in the extreme south. To identify the form of this upper curve, while conserving fold class, dip isogons are projected from q_1 and q_2 down to the reference-curve. The orthogonal thicknesses t_1 and t_2 and the isogon orientations ϕ_1 and ϕ_2 are measured. Knowing α_1, t_1, α_2 and t_2, two very similar values of t_0 are found from (a). The magnitude of the difference between the t_0 values is a measure of the local suitability of the regional curves in (a) and (b). The mean value of t_0 is used to determine values of t_α from (a) for all values of α. Values of ϕ_α are determined from (b). (d) Dip isogons are projected from the reference curve, along which α is readily determined. The isogons lie at angles ϕ_α to the normals from the reference curve and have length $t_\alpha \sec \phi_\alpha$. Their ends lie on an interpolating curve through q_1 and q_2, which together with the reference curve produces a layer which conforms with all the data from (a), (b) and (c).

1973) from smaller scale or laterally equivalent folds, then interpolations can be made which conserve this fold class (Fig. 7). This is achieved by first constructing a 'reference-curve', using rotated cubic interpolation, for the layer trace which has most control (Fig. 7c.). Strictly, the reference curve could be drawn by any interpolation method, but rotated cubics are mathematically very simple, permitting the first derivative (slope) to be easily determined at any point along the arc. Having constructed the reference curve, dip isogons can then be drawn from points q_1 and q_2 on an adjacent trace to it, permitting values of t_α (the orthogonal thickness) and ϕ_α (the angle between an isogon and the normal to the reference curve) to be determined for the corresponding dips (α_1 and α_2) at q_1 and q_2. Knowing the curve for the limb on a t'_α plot, and an orthogonal thickness t_α for a given dip, then a value for t_0 can be determined. Using t_0, values of both t_α and ϕ_α can be read from the plots for all values of α and a smooth curve constructed through q_1 and q_2 by projecting dip isogons of length $t_\alpha \sec \phi_\alpha$ from the reference curve.

If the t'_α curves exactly represented all the folds in a region, then the values for t_0 derived from q_1 and q_2 would be the same, but in practical applications they would differ slightly. Two values of t_0 would produce two close but distinct curves, one through q_1 and the other through q_2, which would fail to meet at the hinge. This may be overcome by using the mean of the two values of t_0 to ensure that the curves through q_1 and q_2 have coincident hinges. The magnitude of this error in t_0 is dependent on the local suitability of the regional t'_α and ϕ_α curves.

Unlike the Busk construction, rotated cubic interpolation is therefore not restricted to Class 1B folds. If data concerning fold class are available, the class can be conserved in the interpolation.

PRACTICAL APPLICATION OF THE INTERPOLATION TECHNIQUES

By sequentially rotating the reference frame, cubic interpolators become as flexible and simple as their physical analogue, the draughtsman's spline. Through a computer, digitising tablet and plotter, section drawing and restoring can become highly automated, precise and reproducible, whilst fully and conservatively utilising the data. It is, however, very important to recognise that such techniques must work alongside standard geological extrapolation and interpolation methods to make full use of the data. Such methods include projection of stratigraphic thicknesses, extrapolation of fold axes, stereographic orientation analysis and determination of the degree of thickening at hinges. The flexibility and compatibility of the rotated cubic arcs allows them to be used as an integral part of a totally objective section drawing package. Drawing a subjective free-hand profile wastes objectively argued controls.

To test the accuracy of the rotated cubic interpolations the position and slopes of 40 points on six layer traces in a photograph from Weiss (1972, plate 84 A) were sampled (Fig. 8). Both the bisecting arcs and the arcs utililsing axial-trace data are marginally less conservative than the minimum arc interpolation, but on the scale of the reproduction are practically indistinguishable. The interpolation is reassuringly accurate, whilst consistently tending to conservatism. A tighter reproduction could have been achieved by constraining the curves through control at the hinges, but the object of this excercise is to illustrate that with minimal control (an average of only 2.5 data points per hinge) a very satisfactory interpolation can be made. A measure of the

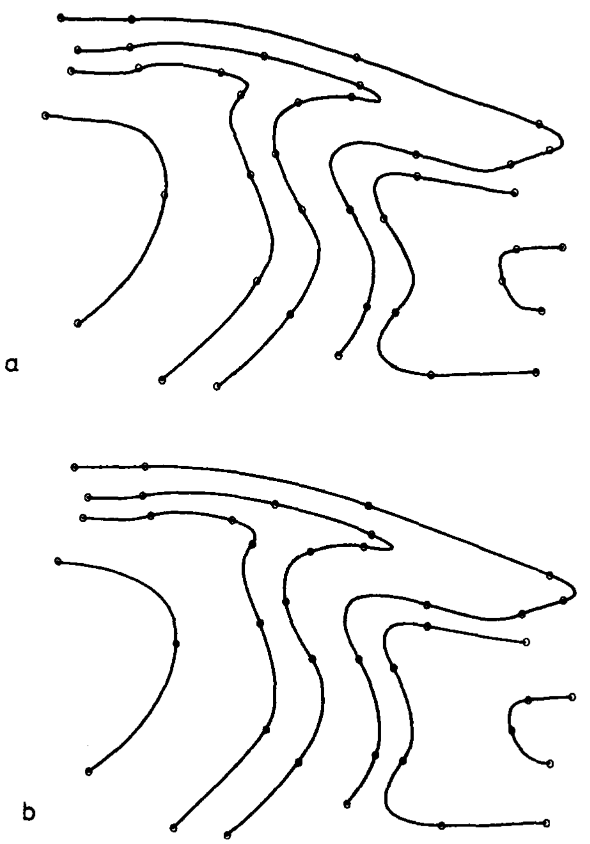

Fig. 8. (a) Layer traces of folded graywackes taken from Weiss (1972, plate 84 A). The location and slopes of the layers were sampled at 40 points (open circles). (b) The minimum arc-length interpolation through the data. The sum of the lengths of the interpolated traces is within 1.5% of the original with a data density as low as 2.5 points per hinge.

accuracy of the interpolation is that the sum of the lengths of the interpolated layer traces is within 1.5% of the original.

CONCLUSIONS

Practices such as free-hand curve drawing squander objectively gained data, leading to wide variations in restored bed length. Cubic splines in their basic form are of little practical use in section drawing because they cannot cope with overturned layers. Parametric interpolation can cope, but is most appropriate where the data are discrete closely-spaced points. To utilise the common data format of apparent dips in the section, while coping with overturned layers, a new set of methods involving rotation of the reference frame allows accurate though slightly conservative interpolations to be made. The necessary calculations in the method are far simpler than those involved in the basic or parametric approaches, making its use simple.

The first method, using vertical ordinates, is found to produce a very poor interpolation. The other five methods, however, are very successful. When the ordinate is rotated so that it bisects consecutive dips, a conservative and realistic curve is produced which is usually very close in morphology and length to minimum arc-length cubic splines. This would permit the former (which is computationally quicker) to substitute the latter in many instances. Minimum arc-length curves are very useful for establishing an objective lower limit to strain estimates in balanced sections. Additional data (e.g. the orientation and location of axial traces, additional location points, and the fold class) can be fully exploited to constrain the orientation of the interpolation ordinate and thus the length and morphology of the curve. In the example presented, rotating w, so that it (i) bisects consecutive apparent dips, (ii) generates a minimum arc-length curve and (iii) conforms with axial trace data, all produce very similar profiles which differ in arc-length from the original by less than 1.5%. This was produced with data densities as low as 2.5 points per hinge.

This set of interpolation techniques should be used as an integral part of a totally objective section drawing package, fully utilising standard section-construction techniques such as down-plunge and layer-normal projection of stratigraphic thicknesses, stereographic orientation analysis and degree of layer thickening or thinning. The methods presented here should be used in the final stages of section drawing, for systematically interpolating between all the argued controls while maintaining the objectivity of the data.

Acknowledgements—The author thanks D. J. Sanderson (The Queen's University, Belfast) for valuable advice and criticism. The research was supported by a studentship from BP Petroleum Development Ltd.

REFERENCES

Busk, H. G. 1929. *Earth Flexures*. Cambridge Univ. Press, London.
Dahlstrom, C. D. A. 1969. Balanced cross-sections. *Can. J. Earth Sci.* **6**, 743–757.
Evans, D. G., Schweitzer, P. N. & Hanna, M. S. 1985. Parametric cubic splines and geologic shape descriptions. *J. Int. Ass. math. Geol.* **17**, 611–624.
Hudleston, P. J. 1973. Fold morphology and some geometrical implications of theories of fold development. *Tectonophysics* **16**, 1–46.
Johnson, L. W. & Riess, R. D. 1982. *Numerical Analysis*. Addison–Wesley, Reading, MA.
Marchuk, G. I. 1982. *Applications of Mathematics*: 2: *Methods of Numerical Mathematics*. Springer, New York.
Ramsay, J. G. 1967. *Folding and Fracturing of Rocks*. McGraw-Hill, New York.
Schoenberg, I. J. 1946. Contributions to the problem of approximation of equidistant data by analytic functions. *Q. appl. Math.* **4**, 45–99.
Weiss, L. E. 1972. *The Minor Structures of Deformed Rocks: A Photographic Atlas*. Springer, New York.

APPENDIX 1: GENERAL ALGORITHM FOR DETERMINING POINTS ON THE INTERPOLATING CURVE

1. Input data pair (x_1, z_1, α_1) and (x_2, z_2, α_2).
2. Input θ so that:
   (i) w bisects α_1 and α_2;
   (ii) interpolation is compatible with the axial trace;
   (iii) interpolation is compatible with borehole data, etc.;
   (iv) minimum length arc can be found; that is, input an appropriate range of θ, such that w approximately bisects α_1 and α_2 and test when the first coefficient $a \rightarrow 0$.

3. Translate data pair to local origin (x_1, z_1):

$$\begin{pmatrix} x_i \\ z_i \\ \alpha_i \end{pmatrix} \rightarrow \begin{pmatrix} x_i' \\ z_i' \\ \alpha_i' \end{pmatrix}$$

$$x_i' = x_i - x_1,$$
$$z_i' = z_i - z_1,$$
$$\alpha_i' = \alpha_i,$$
$$(i = 1, 2).$$

4. Rotate data pair through an angle θ:

$$\begin{pmatrix} x_i' \\ z_i' \\ \alpha_i' \end{pmatrix} \rightarrow \begin{pmatrix} x_i'' \\ z_i'' \\ \alpha_i'' \end{pmatrix}$$

$$x_i'' = x_i' \sin\theta - z_i' \cos\theta,$$
$$z_i'' = x_i' \cos\theta + z_i' \sin\theta,$$
$$\alpha_i'' = \alpha_i' + 90 - \theta,$$
$$(i = 1, 2).$$

5a. Determine the coefficient a of the interpolating cubic arc $z'' = a(x'')^3 + b(x'')^2 + cx''$ (see Appendix 2):

$$a = \frac{(\tan\alpha_1'' + \tan\alpha_2'')\, x_2'' - 2z_2''}{(x_2'')^3}$$

(minimum arc when $a \rightarrow 0$).

5b. Determine the remaining coefficients:

$$b = \frac{3z_2'' - (2\tan\alpha_1'' + \tan\alpha_2'')\, x_2''}{(x_2'')^2},$$

$$c = \tan\alpha_1''.$$

6. Sample arc by substituting values of x'' at regular intervals between $x'' = x_1'' = 0$ and $x'' = x_2''$.

7. Reverse-rotate sampled (x'', z'') co-ordinates:

$$\begin{pmatrix} x'' \\ z'' \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ z' \end{pmatrix}$$

$$x' = x'' \sin\theta + z'' \cos\theta,$$
$$z' = -x'' \cos\theta + z'' \sin\theta.$$

8. Reverse-translate the (x', z') co-ordinates:

$$\begin{pmatrix} x' \\ z' \end{pmatrix} \rightarrow \begin{pmatrix} x \\ z \end{pmatrix}$$

$$x = x' + x_1,$$
$$z = z' + z_1.$$

9. The output (x, z) co-ordinates now lie in the data reference frame on an interpolated cubic arc which has been rotated so that its ordinate is θ anticlockwise from horizontal.

10. The length of the arc is given by the standard integral

$$L = \int \sqrt{\{1 + (dz''/dx'')^2\}}\, dx'',$$

which can readily be approximated by the Trapezoidal or Simpson's rule.

APPENDIX 2: DETERMINING THE COEFFICIENTS a, b and c OF THE INTERPOLATING ARC

After translating and rotating the data pair, the following four equations apply:

$$z_1'' = a(x_1'')^3 + b(x_1'')^2 + cx_1'' + d,$$
$$\tan\alpha_1'' = 3a(x_1'')^2 + 2bx_1'' + c,$$
$$z_2'' = a(x_2'')^3 + b(x_2'')^2 + cx_2'' + d,$$
$$\tan\alpha_2'' = 3a(x_2'')^2 + 2bx_2'' + c.$$

Since (x_1'', z_1'') lies at the origin the first two equations simplify to:

$$d = 0$$
$$\text{and } c = \tan\alpha_1''.$$

Solving the last two simultaneously gives:

$$a = \frac{(\tan\alpha_1'' + \tan\alpha_2'')x_2'' - 2z_2''}{(x_2'')^3}$$

$$b = \frac{3z_2'' - (2\tan\alpha_1'' + \tan\alpha_2'')x_2''}{(x_2'')^2}.$$